

The Scottish Mathematical Council: Mathematical Challenge 2013-14
Solutions to Middle Division: Problems 1

M1.1

In a recent election, six candidates stood and a total of 51880 votes were cast. The winning candidate beat the others by 1336, 7085, 15333, 15654 and 17102 votes respectively.

Candidates lose their deposit if they fail to get more than 5% of the total number of votes cast.

How many candidates lost their deposits?

Solution

Let v be the number of votes obtained by the winner. So the total number of votes cast was

$$v + (v - 1336) + (v - 7085) + (v - 15333) + (v - 15654) + (v - 17102) = 6v - 56510.$$

The total number of votes cast was 51880. Hence

$$6v - 56510 = 51880$$

i.e. $6v = 56510 + 51880 = 108\,390$

i.e. $v = 18065.$

So the six candidates got the following numbers of votes:

$$18065$$

$$18065 - 1336 = 16729$$

$$18065 - 7085 = 10980$$

$$18065 - 15333 = 2732$$

$$18065 - 15654 = 2411$$

$$18065 - 17102 = 963.$$

As 5% of the total is 2594, two candidates each lost their deposit.

M1.2

In a tennis tournament, each match is played between two players, and the winner proceeds to the next round whereas the loser is eliminated. There are no draws.

If necessary, in the first round only, a number of players do not participate.

(a) A particular tournament starts with 256 players and proceeds until there is one overall winner. How many matches are played in this tournament?

(b) If the tournament starts with 296 players and proceeds until there is one overall winner, how many matches are played in this tournament?

Solution

(a) Starting with 256 players, if all play in the 128 first round matches, the number of matches in successive rounds will be 64, 32, 16, 8, 4, 2, 1.

$$\text{Number of matches played} = 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = 255 \text{ matches}$$

(b) In this case there are 40 more players than in (a) so there must be 40 more matches. Thus there are 295 matches.

Alternative Solution

(a) Each match eliminates one player.

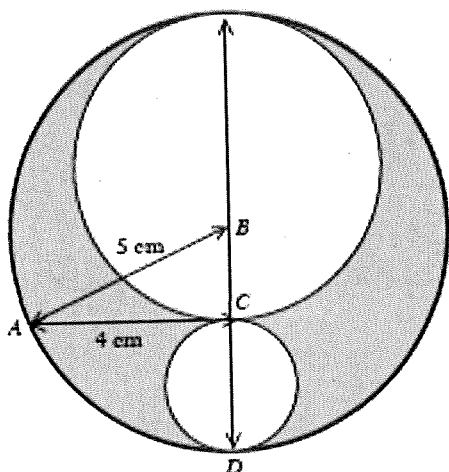
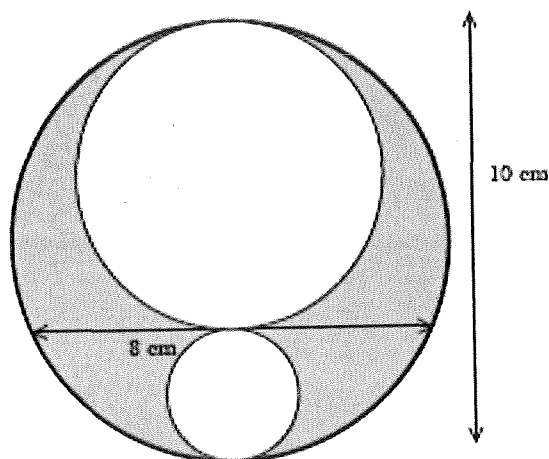
So to eliminate 255 players 255 matches are required.

(b) 295 matches

M1.3

A jeweller has been asked to make a pendant in the shape of the shaded area shown. The height of the pendant is 10 cm and the distance across at the point where the two smaller circles touch is 8 cm. Find the area of the pendant.

Solution



Radius of large circle = 5 cm so $BD = BA = 5$.

Half of the length of the chord = 4 cm so $AC = 4$

Applying Pythagoras' Theorem to $\triangle ABC$ gives $BC = 3$ so the radius of the middle circle = 4 cm and $CD = 2$.

Radius of smallest circle = 1 cm.

Shaded area

$$= \pi (5^2 - 4^2 - 1^2) = 8\pi \text{ cm}^2$$

$$\approx 25.13 \text{ cm}^2$$

M1.4

On a tiny remote island where the death sentence still exists a man can be granted mercy after receiving the death sentence in the following way:

- he is given 18 white balls and 6 black balls.

- he must divide them between three boxes with at least one ball in each box.
- he is then blindfolded and must choose a box at random and then a single ball from within this box.

He receives mercy only if the chosen ball is white.

Find the probability that he receives mercy when he distributes the balls in the most favourable manner.

Solution

If any box contains only black balls, then the probability of reprieve is at most $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

There are configurations with a greater probability. For instance, distributing the balls evenly (6w and 2b in each box) gives a probability of $\frac{3}{4}$.

Thus he certainly wants to have at least one white ball in each box. For a particular box, the contribution to the overall probability is greatest when it contains only white balls. In that case, if it contains more than one ball then all but one of these can be transferred to one of the other boxes that contains a black ball to further improve the chance of a reprieve.

This suggests that the best strategy is to put a single white ball in each of two of the boxes and all the remaining balls (16w and 6b) in the third box.

The probability is then $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \times \frac{16}{22} = \frac{10}{11}$

(See separate sheet for a more complete justification)

M1.5

Three types of item, A, B and C, are for sale. Items of type A sell at 8 for £1. Items of type B sell for £1 each. Items of type C sell for £10 each. A selection of 100 items which includes at least one of each type costs £100. How many items of type B are there in the selection?

Solution

Let a be the number of items of type A in the selection.

Let b be the number of items of type B in the selection.

Let c be the number of items of type C in the selection.

$$\frac{a}{8} + b + 10c = 100$$

$$a + b + c = 100$$

$$\text{So } b = 100 - a - c$$

$$\frac{a}{8} + 100 - a - c + 10c = 100$$

$$- \left(\frac{7}{8}\right)a + 9c = 0$$

$$7a = 72c$$

Either: $a = c = 0$ (which would mean that there were item types missing from the selection, and hence is not possible)

Or: $a = 72$ and $c = 7$ (larger multiples of these are not possible as there are only 100 items in all.)

$$\text{Hence } b = 100 - 72 - 7 = 21$$

So there are 21 items of type B.

More complete proof, for your interest!

Solution to problem 51.1/M1.4

Firstly, notice that, if any box contains only black balls, then the probability of reprieve is less than $1/3 + 1/3 = 2/3$. There are configurations with a greater probability (e.g. distributing both colours equally gives a probability of $3/4$). We can therefore assume that each box contains at least one white ball.

Now suppose that two of the boxes contain at least one black ball. Let the first contain x white balls and a total of y balls and the second s white balls and a total of t balls. Then our assumptions are that $1 \leq x < y$ and $1 \leq s < t$ and the contribution of these two boxes to the probability of a reprieve is

$$\frac{1}{3} \times \left\{ \frac{x}{y} + \frac{s}{t} \right\}.$$

Now transfer all but one white and all the blacks from the first to the second box. The contribution from these boxes is now

$$\frac{1}{3} \times \left\{ 1 + \frac{s+x-1}{t+y-1} \right\}.$$

The contribution from the third box is unaltered. Now

$$\begin{aligned} 1 + \frac{s+x-1}{t+y-1} - \left\{ \frac{x}{y} + \frac{s}{t} \right\} &= \frac{yt(t+y-1) + yt(s+x-1) - xt(t+y-1) - ys(t+y-1)}{yt(t+y-1)} \\ &= \frac{y^2(t-s) + t^2(y-x) + xt + sy - 2yt}{yt(t+y-1)} \\ &\geq \frac{y^2 + t^2 + xt + sy - 2yt}{yt(t+y-1)} \\ &= \frac{(y-t)^2 + xt + sy}{yt(t+y-1)} > 0. \end{aligned}$$

Here we have used the fact that $t-s \geq 1$ and $y-x \geq 1$. This shows that, by transferring as above, the probability of reprieve increases.

It follows that, in the optimal configuration, two of the boxes will contain only white balls. If either contains two or more white balls, the transfer of all but one of these to the third box will further increase the probability of a reprieve. This means that the optimal configuration is when two of the boxes contain one white and no black balls, whilst the third contain 16 white and 6 black balls. The resulting probability is

$$\frac{1}{3} \times \left\{ 1 + 1 + \frac{16}{22} \right\} = \frac{10}{11}.$$